# The displacement effect of a sphere in a two-dimensional shear flow

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## Summary

This paper contains a theoretical investigation of the displacement effect of a pitot tube in a shear flow. Viscosity is neglected throughout so that the vorticity field alone is considered.

It is first shown that a two-dimensional approach does not produce a large enough displacement effect because it does not include the stretching of vortex tubes that takes place around a three-dimensional pitot tube. Then the three-dimensional problem is considered. A solution is obtained in the plane of symmetry for a sphere in a shear flow. This solution is found by making an assumption about the rate of stretching of vortex tubes perpendicular to the plane of symmetry and then considering the shear flow as a small perturbation of a uniform flow. A solution in the plane of symmetry is sufficient to obtain the displacement effect, which is found to be of the same order as the experimental result obtained by Young & Maas (1936) for a conventional pitot tube. The sphere may be considered to represent a conventional pitot tube (of slightly smaller diameter), so it is concluded that a large part of the displacement effect of a pitot tube may be accounted for without the inclusion of viscosity, i.e. by consideration of the vorticity field alone.

To a first approximation, the vorticity in the plane of symmetry is found to depend only on the distance from the centre of the sphere.

An outline of shear flows past some two-dimensional bodies is given in an appendix. The bodies considered are a circular cylinder and a two-dimensional 'pitot-tube' consisting of two parallel semi-infinite plates.


## 1. Introduction

Correct interpretation of the pressure measured by a pitot tube in a shear flow is important in the experimental study of boundary layers and wakes, whose velocity profiles are generally obtained from pitot traverses. The problem has been investigated experimentally by Young & Maas (1936) and their approach, outlined below, shows the nature of the problem. The corresponding theoretical problem is then discussed, and an approximate solution is obtained for a spherical 'pitot tube'.

The variation of total pressure across a wake, measured by a pitot tube, is shown in figure 1, together with the hypothetical curve which would be obtained using a tube of zero diameter. In effect, the pitot tube with its axis at $B$ measures the total pressure at $A$. Hence, if the distance $\delta$ were known, the actual distribution could be obtained from the measured one. The distance $\delta$ must be a function of the shape of the pitot tube, its diameter $D$, the velocity and its derivatives, and the kinematic viscosity.

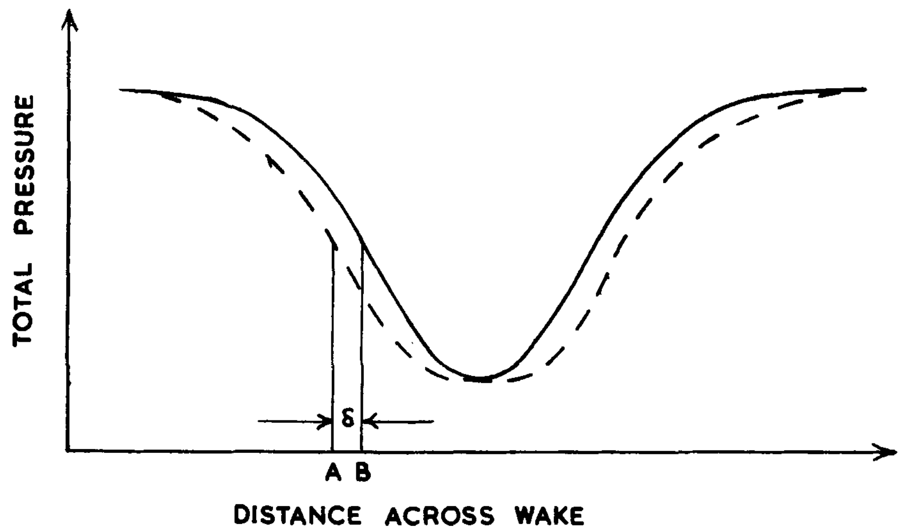


Figure 1. The variation of total pressure across a wake, showing displacement effect due to finite size of pitot tube : ——— measured distribution; ------ actual distribution.

Suppose now that the second and higher derivatives of the velocity are zero, or so small that they can be neglected; i.e. the velocity of the oncoming flow is

$$U + Ay', \tag{1.1}$$

where $U$ is the velocity on the axis of the pitot tube, and $y'$ is a rectangular coordinate perpendicular to the axis. Then, if we consider only geometrically similar tubes, in an inviscid fluid,

$$\delta = \text{function of } D,\ U,\ A, \tag{1.2}$$

or, from dimensional considerations,

$$\frac{\delta}{D} = \text{function of } K, \tag{1.3}$$

where

$$K = \frac{AD}{2U}. \tag{1.4}$$

Using flat-nosed pitot tubes with internal diameters approximately equal to 0·6 of their external diameters, Young & Maas found that

$$\frac{\delta}{D} = 0{\cdot}18\,\mathrm{sgn}\,K \tag{1.5}$$

i.e. the displacement is towards the region of higher velocity. This result contains a difficulty at $K = 0$, where there is a discontinuity in the displacement effect. This is not physically plausible since it implies that there is a discontinuous change in the flow pattern when the pitot tube is moved a small distance from a point of zero shear. However, the result obtained by Young & Maas is not necessarily valid near $K = 0$. Their values of $\delta/D$ show increasing scatter as $K$ decreases, and no values were obtained for $K$ less than 0·025. Also, when $K$ is small, it is probably not always justified to neglect second and higher derivatives of the velocity. Therefore, it is likely that the actual variation of $\delta/D$ with $K$ is rapid, but finite, when $K$ is very small, and that $\delta/D$ tends towards some value, constant for a particular pitot tube geometry (0·18 for the shape of tube used by Young & Maas), at larger values of $K$. During recent work at Cambridge University, further experimental values have been obtained for the displacement effect which are in fair agreement with the Young & Maas result.

So far, no theoretical treatment of a pitot tube in a shear flow has been made. Two classes of problems have been solved in which the oncoming flow has a uniform rate of shear:

(*a*) those in which the body is cylindrical and the whole flow is two-dimensional (Lamb 1932, p. 233, Lighthill, Reichardt 1954, Tsien 1943); some of these solutions are outlined in an appendix;

(*b*) those in which the body is cylindrical with generators perpendicular to the oncoming vorticity (Hawthorne 1954, Squire & Winter 1951).

Other solutions involving bodies in non-uniform streams have been obtained, but these are not relevant to the pitot tube problem (e.g. Nagamatsu 1951, Hawthorne & Martin 1955).

In the completely two-dimensional problem, the displacement effect can be found easily, but it is considerably smaller than the experimental values obtained with an axially-symmetric pitot tube. This is not necessarily due to the omission of viscosity (Reichardt's experiments show very good agreement with inviscid theory) but is probably due to an essential difference between the vorticity fields in shear flows past two and three-dimensional bodies. Thus, consider a vortex tube moving towards the pitot tube. At a large distance upstream, the vorticity is everywhere perpendicular to both the velocity and the direction of the shear, so that a vortex tube is initially a cylinder with its generators perpendicular to the stream direction. When the flow is completely two-dimensional, the generators remain perpendicular to the stream, so that the cross-section of a vortex tube is deformed but its area is unaltered. Hence the vorticity is constant along any streamline. In front of a three-dimensional object, however, there is a retardation of the fluid, while at large lateral distances the flow is almost unaffected by the body. Therefore, the vortex tube must be stretched so that there is a decrease in cross-sectional area (by continuity) and hence an increase in vorticity (by Kelvin's theorem). The net result is an accumulation of vorticity in front of the body and, in consequence, larger streamline curvatures than occur in the two-dimensional case. This makes possible a much larger displacement effect. For example, it is shown in this paper that, when the vorticity of the oncoming flow is small, a sphere produces, according to certain approximate assumptions, a displacement approximately five times as large as that produced by a cylinder of the same radius.

Another feature introduced by the stretching of the vortex tubes around a three-dimensional object is a streamwise component of vorticity. In the plane of symmetry, the velocity field of this component seems to be predominantly downwards, which would tend to increase the displacement effect. This feature is not accounted for explicity in the analysis of this paper, and would have to be taken into account in any attempt to improve the present theory.

The three-dimensional problem is much more difficult, because the full equations are complicated and non-linear. However, the streamline reaching the stagnation point must always lie in the plane of symmetry, $z = 0$, so that a solution in this plane is sufficient to determine the displacement effect. The flow pattern in this plane is not independent of the rest of the flow pattern, but it can be shown to depend on one quantity only, namely $(\partial w/\partial z)_{z=0}$, where $w$ is the velocity in the $z$-direction. As a first attack on the problem, the equations of motion in the plane of symmetry are solved below by assigning an approximate value to $(\partial w/\partial z)_{z=0}$. Consider again a vortex tube moving towards the body. It is clear that the behaviour of the flow pattern in the plane of symmetry is governed by the rate of stretching of the vortex tube perpendicular to this plane. This quantity is the relevant component of the rate of strain tensor, i.e. $\partial w/\partial z$. Therefore, it is necessary to know $(\partial w/\partial z)_{z=0}$ in order to solve the problem in the plane of symmetry. This is not possible without a full solution of the three-dimensional problem, so an approximate value is sought. Since the vorticity in the plane of symmetry is in the $z$-direction, it seems reasonable to assume that, in this plane, $\partial w/\partial z$ is independent of the vorticity. On this assumption, $(\partial w/\partial z)_{z=0}$ must take its value in the irrotational flow.

At this stage the problem is still non-linear: in fact, it involves the solution of three simultaneous partial differential equations. However, in practice the oncoming shear is often small, which suggests that linearization may be effected by considering the shear flow as a perturbation of the uniform flow. Since small shear corresponds to small $K$, a solution is sought in the form of a series in ascending powers of $K$. To be able to use this approach, it is necessary to know the solution when $K$ is zero. This solution is not available for a pitot tube of conventional shape but only for bodies with closed, rounded noses. Therefore, since the solution for a sphere in a uniform flow is simple and well known (e.g. Milne-Thomson 1949, p. 413), an attempt has been made to obtain a solution for a sphere in a shear flow.

If the sphere itself were used as a pitot tube, by making a small pressure hole on the axis, the definition of $\delta$ as the displacement of the stagnation streamline would not have the desired physical meaning. The pressure measured would be the static pressure at the orifice, which is less than the total pressure on the stagnation streamline. However, a pitot tube of this type would be unsatisfactory because of its sensitivity to yaw. Instead, one may think of the sphere as replacing a conventional pitot tube of slightly smaller outside diameter, so that the streamline reaching the stagnation point is the same as the one reaching the stagnation point inside the pitot tube. Then the displacement effect of the sphere is defined in the same way as for an open body; i.e. the asymptotic distance of the stagnation streamline from the axis. The above argument is illustrated by the two-dimensional case (see the appendix) in which the displacement effects of a cylinder and a parallel plate pitot tube are equal when the diameter of the cylinder is $\sqrt{2}$ times the spacing of the plates.

By expanding the full equations of motion in powers of $K$, it can be shown that the assumption for $(\partial w/\partial z)_{z=0}$ gives the vorticity correct to the first order in $K$, but omits an irrotational contribution to the velocity components. Unfortunately, this cannot be remedied without solving the problem for the entire three-dimensional flow pattern*. However, in order to obtain other than a linear variation for the displacement effect, results have been obtained up to the order of $K^3$ using the above assumption for $(\partial w/\partial z)_{z=0}$. The cubic term produces a maximum value of $\delta/D$ which is of the same order as the asymptotic values found experimentally and which occurs within the possible working range of values of $K$.

The result for the first approximation to the vorticity is of some interest since it shows that it is a function of the distance from the centre of the sphere only, and that it has a square root infinity on the surface. In fact, the first approximation is $(1-r^{-3})^{-1/2}$ times the oncoming vorticity, where $r$ is the distance from the centre of the sphere divided by its radius.

The mathematical formulation of the problem, using the above assumptions, is given in § 2, and the equations are solved in § 3. The solution is summarized in § 4, and the velocity components are obtained. In § 5, differential equations for the streamlines are found and solved for the stagnation streamline. Outlines of some two-dimensional solutions are given in an appendix.

## 2. Mathematical formulation of the problem

Consider a cylindrical coordinate system $r'$, $\theta$, $z'$ with its origin at the centre of a sphere of radius $a$. The undisturbed flow is independent of $z'$, so that $z' = 0$ is the plane of symmetry of the flow pattern, and is in the direction $\theta = \pi$ (figure 2). Non-dimensional coordinates $r$, $z$, are defined as $r'/a$, $z'/a$.

* Since this work was completed, Lighthill has solved the first order problem over the entire flow field. He obtains the same order of magnitude for the displacement effect and for the downwash velocity ahead of the sphere. This increases one's confidence in the cubic term obtained in this paper.

In the plane of symmetry, the vorticity components are $(0, 0, \zeta)$ and the velocity is $(u, v, 0)$ where $u$ and $v$ are even functions of $z$. Therefore, in this plane, Helmholtz's equations of motion reduce to

$$u\frac{\partial \zeta}{\partial r} + v\frac{\partial \zeta}{r\partial \theta} = \zeta\left(\frac{\partial w}{\partial z}\right)_{z=0}, \tag{2.1}$$

where
$$\frac{1}{r}\frac{\partial (rv)}{\partial r} - \frac{\partial u}{r\partial \theta} = a\zeta. \tag{2.2}$$

The equation of continuity is

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial v}{r\partial \theta} = -\left(\frac{\partial w}{\partial z}\right)_{z=0}. \tag{2.3}$$

Equations (2.1) to (2.3) are the same as those for a two-dimensional rotational flow except for the additional terms containing $(\partial w/\partial z)_{z=0}$. If a value is inserted for $(\partial w/\partial z)_{z=0}$, the problem is reduced, in effect, to a two-dimensional

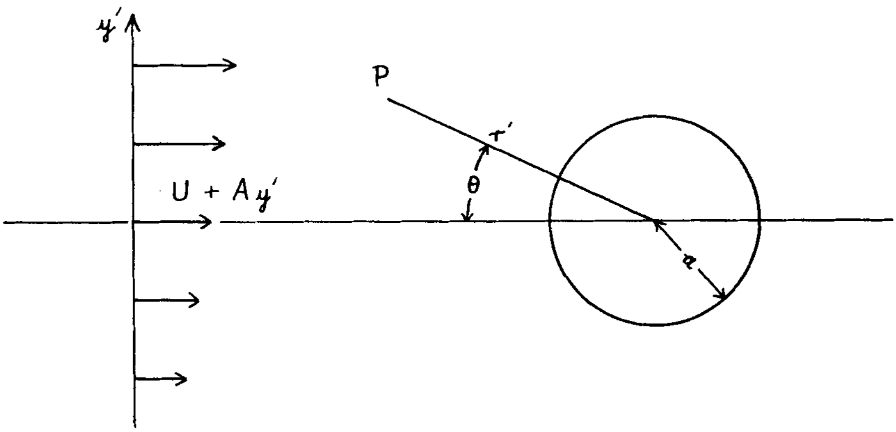


Figure 2. The plane of symmetry of the flow pattern.

one, and the above equations are sufficient to determine $u$, $v$, and $\zeta$, provided that the boundary conditions are suitable. In § 1, it was shown that a good approximation to $(\partial w/\partial z)_{z=0}$ is its value in the irrotational flow. Therefore (Milne-Thomson 1949),

$$\left(\frac{\partial w}{\partial z}\right)_{z=0} = \frac{3U\cos\theta}{2r^4}. \tag{2.4}$$

By means of the transformation

$$\left.\begin{aligned} u &= U\frac{\partial \phi}{\partial r} + Aa\frac{\partial \psi}{r\partial \theta} = U\left(\frac{\partial \phi}{\partial r} + K\frac{\partial \psi}{r\partial \theta}\right), \\ v &= U\frac{\partial \phi}{r\partial \theta} - Aa\frac{\partial \psi}{\partial r} = U\left(\frac{\partial \phi}{r\partial \theta} - K\frac{\partial \psi}{\partial r}\right), \end{aligned}\right\} \tag{2.5}$$

equations (2.2) and (2.3) become

$$\nabla^2\phi = -\frac{3\cos\theta}{2r^4}, \tag{2.6}$$

$$\nabla^2\psi = -\frac{\zeta}{A}, \tag{2.7}$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{2.8}$$

The oncoming flow is parallel and has uniform rate of shear. Hence, the boundary conditions are

$$\begin{aligned}
&(a)\ \text{at } r = \infty, && \zeta = A,\\
& && -u = U\cos\theta + Aar\sin\theta\cos\theta,\\
& && v = U\sin\theta + Aar\sin^2\theta,\\
&(b)\ \text{at } r = 1, && u = 0.
\end{aligned}$$

However, when $A$ is zero, the solution is known (Milne-Thomson 1949, p. 413). Thus, $\phi$ satisfies the equations

$$\left.\begin{aligned}
\frac{\partial\phi}{\partial r} &= -\cos\theta\left(1-\frac{1}{r^3}\right),\\
\frac{\partial\phi}{r\partial\theta} &= \sin\theta\left(1+\frac{1}{2r^3}\right).
\end{aligned}\right\} \tag{2.9}$$

But this solution for $\phi$ satisfies (2.6) when $A$ is not zero.* Hence the problem is reduced to that of solving (2.1) and (2.7) with boundary conditions

$$\left.\begin{aligned}
&(a)\ \text{at } r = \infty, && \zeta = A,\\
& && \psi = -\tfrac{1}{2}r^2\sin^2\theta,\\
&(b)\ \text{at } r = 1, && \psi = \text{constant}.
\end{aligned}\right\} \tag{2.10}$$

The solution is obtained by inserting expansions for $\zeta$ and $\psi$ in powers of $K$. These are

$$\left.\begin{aligned}
\psi &= \psi_0 + K\psi_1 + K^2\psi_2 + \dots,\\
\frac{\zeta}{A} &= \zeta_0 + K\zeta_1 + K^2\zeta_2 + \dots.
\end{aligned}\right\} \tag{2.11}$$

On substituting (2.11) in (2.1) and (2.7) two series of differential equations are obtained. These are

$$\frac{\partial\phi}{\partial r}\frac{\partial\zeta_n}{\partial r} + \frac{\partial\phi}{r\partial\theta}\frac{\partial\zeta_n}{r\partial\theta} = \frac{3\cos\theta}{2r^4}\zeta_n + g_n(\psi_0, \dots \psi_{n-1}, \zeta_0, \dots \zeta_{n-1}), \tag{2.12}$$

$$\nabla^2\psi_n = -\zeta_n, \tag{2.13}$$

* It would not do so if the term of order $K$ in $(\partial w/\partial z)_{z=0}$ were included.

where the functions $g_n$ are given by

$$\left.\begin{aligned} g_0 &= 0, \\ g_n &= \sum_{i+j=n-1} \left(\frac{\partial\psi_i}{\partial r}\frac{\partial\zeta_j}{r\partial\theta} - \frac{\partial\psi_j}{r\partial\theta}\frac{\partial\zeta_i}{\partial r}\right). \end{aligned}\right\} \tag{2.14}$$

The boundary conditions are

$$\left.\begin{aligned} &(a)\ \text{at } r=\infty, \quad \zeta_0 = 1, \quad \psi_0 = -\tfrac{1}{2}r^2\sin^2\theta, \\ &\qquad\qquad\qquad\ \ \zeta_n = 0, \quad \psi_n = 0, \quad (n \neq 0), \\ &(b)\ \text{at } r=1, \quad \psi_n = \text{constant}. \end{aligned}\right\} \tag{2.15}$$

In § 3, equations (2.12) and (2.13) are solved for $n = 0$, 1, and 2. First, equation (2.12) with $n = 0$ is solved to obtain the first approximation to the vorticity. This is substituted in (2.13) to obtain the first approximation to $\psi$. Then, substituting back in (2.12) with $n = 1$, the second approximation to the vorticity is found. This is inserted in (2.13), and so on. Physically, this corresponds to finding first the vorticity field obtained by convecting initially constant vorticity along the irrotational streamlines, and then the velocity perturbations caused by this vorticity field. On substituting back into (2.12), the vorticity field that is convected along the perturbed streamlines is found. Then the further velocity perturbation is obtained, and the process is continued until the desired approximation is reached.

## 3. Solution of the differential equations

The general solutions of equations (2.12) and (2.13) are found below and then the special solutions satisfying the boundary conditions (2.15) are determined.

Substituting from (2.9), equation (2.12) is

$$-\cos\theta\left(1-\frac{1}{r^3}\right)\frac{\partial\zeta_n}{\partial r} + \sin\theta\left(1+\frac{1}{2r^3}\right)\frac{\partial\zeta_n}{r\partial\theta} = \frac{3\cos\theta}{2r^4}\zeta_n + g_n. \tag{3.1}$$

The general solution of (3.1) is the sum of a particular integral, which depends on $g_n$, and the complementary function $\eta$, which is the solution of

$$-\cos\theta\left(1-\frac{1}{r^3}\right)\frac{\partial\eta}{\partial r} + \sin\theta\left(1+\frac{1}{2r^3}\right)\frac{\partial\eta}{r\partial\theta} = \frac{3\cos\theta}{2r^4}\eta. \tag{3.2}$$

Changing the independent variables from $r$, $\theta$ to $r$, $\chi$, (3.2) becomes

$$-\cos\theta\left(1-\frac{1}{r^3}\right)\left(\frac{\partial\eta}{\partial r}\right)_\chi + \left\{-\cos\theta\left(1-\frac{1}{r^3}\right)\frac{\partial\chi}{\partial r} + \right.$$
$$\left. + \sin\theta\left(1+\frac{1}{2r^3}\right)\frac{\partial\chi}{r\partial\theta}\right\}\left(\frac{\partial\eta}{\partial\chi}\right)_r = \frac{3\cos\theta}{2r^4}\eta. \tag{3.3}$$

Therefore, if $\chi$ is an integral of

$$\frac{dr}{-\cos\theta(1-r^{-3})} = \frac{rd\theta}{\sin\theta(1+\frac{1}{2}r^{-3})}, \tag{3.4}$$

e.g.

$$\chi = \left(\frac{r^3-1}{r}\right)^{1/2}\sin\theta, \tag{3.5}$$

equation (3.3) reduces to

$$-\left(1-\frac{1}{r^3}\right)\left(\frac{\partial\eta}{\partial r}\right)_\chi = \frac{3}{2r^4}\eta, \tag{3.6}$$

whose solution is

$$\eta = j(\chi)\left(\frac{r^3}{r^3-1}\right)^{1/2}, \tag{3.7}$$

where $j$ is an arbitrary function, determined by the boundary condition.

Equation (2.13) is

$$\nabla^2\psi_n = -\zeta_n. \tag{3.8}$$

This is a Poisson-type equation whose general solution is the sum of a particular integral, depending on $\zeta_n$, and the complementary function, which is independent of $\zeta_n$. The latter is a solution of Laplace's equation in two dimensions, and is

$$\tfrac{1}{2}a_0 + \sum_1^\infty [(a_n r^n + b_n r^{-n})\cos n\theta + (c_n r^n + d_n r^{-n})\sin n\theta], \tag{3.9}$$

where $a_n$, $b_n$, $c_n$, $d_n$ are constants determined by the boundary conditions.

*First approximation to the vorticity*

$g_0$ is zero. Therefore, when $n = 0$, the solution of (3.1) is the complementary function alone, given by (3.7), i.e.

$$\zeta_0 = j(\chi)\left(\frac{r^3}{r^3-1}\right)^{1/2}. \tag{3.10}$$

Since $\zeta_0 = 1$ at $r = \infty$,

$$j(\chi) = 1. \tag{3.11}$$

Therefore,

$$\zeta_0 = \left(\frac{r^3}{r^3-1}\right)^{1/2}. \tag{3.12}$$

This equation shows that, to a first approximation, $\zeta$ is independent of $\theta$. This is because, for the particular case of irrotational flow past a sphere, $u$ and $(\partial w/\partial z)_{z=0}$ have the same variation with $\theta$. This can only occur when the body can be represented by a doublet at the origin, since this is the only case in which the velocity potential is the product of two terms, one of which is a function of $\theta$ only and the other is independent of $\theta$.

Equation (3.12) also shows that the vorticity becomes infinite on the sphere. This is to be expected from the previous considerations regarding the stretching of a vortex tube. In fact, the vorticity must become infinite on any body with a stagnation point.

*First approximation to $\psi$*

Substituting (3.12) in (3.8) gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_0}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi_0}{\partial\theta^2} = -\left(\frac{r^3}{r^3-1}\right)^{1/2}. \tag{3.13}$$

A particular integral is $\Psi$, where

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right) = -\left(\frac{r^3}{r^3-1}\right)^{1/2}. \tag{3.14}$$

Hence,

$$r\frac{d\Psi}{dr} = -\int^r x^{5/2}(x^3-1)^{-1/2}\,dx. \tag{3.15}$$

To ensure convergence of the integral as $r \to \infty$, (3.15) is rewritten as

$$r\frac{d\Psi}{dr} = -\tfrac{1}{2}r^2+\int_r^\infty \{x^{5/2}(x^3-1)^{-1/2}-x\}\,dx. \tag{3.16}$$

Integrating again,

$$\Psi = -\tfrac{1}{4}r^2-Q, \tag{3.17}$$

where
$$Q = \int_r^\infty \frac{ds}{s}\int_s^\infty \{x^{5/2}(x^3-1)^{-1/2}-x\}\,dx, \tag{3.18}$$

or, reversing the order of integration,

$$Q = \int_r^\infty \{x^{5/2}(x^3-1)^{-1/2}-x\}\log\left(\frac{x}{r}\right)dx \tag{3.19}$$

$$\sim \tfrac{1}{2}r^{-1} \quad \text{as } r \to \infty. \tag{3.20}$$

The solution of (3.13) is the sum of $\Psi$ and the complementary function (3.9). Hence, applying the boundary conditions (2.15),

$$\psi_0 = \frac{1}{4}\left(r^2-\frac{1}{r^2}\right)\cos 2\theta-\tfrac{1}{4}r^2-Q. \tag{3.21}$$

*Second approximation to $\zeta$*

Substituting (3.12) and (3.21) in (2.14) gives

$$g_1 = g_1'(r)\sin\theta\cos\theta, \tag{3.22}$$

where
$$g_1' = -\tfrac{3}{2}r^{-5/2}(r^4-1)(r^3-1)^{-3/2}. \tag{3.23}$$

Therefore, when $n = 1$, a particular integral of (3.1) is $Z(r)\sin\theta$, where

$$\frac{dZ}{dr}-\frac{Z}{r} = -\left(1-\frac{1}{r^3}\right)^{-1}g_1', \tag{3.24}$$

i.e.
$$Z = -\frac{3r}{2}\int_r^\infty x^{-1/2}(x^4-1)(x^3-1)^{-5/2}\,dx \tag{3.25}$$

$$\sim -\tfrac{1}{2}r^{-2} \quad \text{as } r \to \infty. \tag{3.26}$$

The solution of (3.1) is

$$\zeta_1 = j(\chi)\left(\frac{r^3}{r^3-1}\right)^{1/2} + Z(r)\sin\theta, \tag{3.27}$$

where $\chi$ is given by (3.5). Then, since $\zeta_1 = 0$ at $r = \infty$,

$$j(\chi) = 0, \tag{3.28}$$

and

$$\zeta_1 = Z(r)\sin\theta. \tag{3.29}$$

*Second approximation to $\psi$*

Substituting (3.29) in (3.8) gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2} = -Z(r)\sin\theta. \tag{3.30}$$

A particular integral is $R(r)\sin\theta$, where

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{R}{r^2} = -Z. \tag{3.31}$$

It is convenient, both for algebraic manipulation and for computation, to consider the integrals $R_1$ and $R_2$, where

$$R_1 = r\frac{dR}{dr} - R = -\frac{1}{r}\int^r x^2 Z(x)\,dx \tag{3.32}$$

$$R_2 = r\frac{dR}{dr} + R = -r\int^r Z(x)\,dx. \tag{3.33}$$

The double integrals on the right hand sides of (3.32) and (3.33) can be reduced to single integrals. Integrating by parts, (3.32) becomes

$$R_1 = -\tfrac{1}{4}r^2 Z + \frac{1}{4r}\int^r x^4\frac{d}{dx}\left(\frac{Z}{x}\right)dx, \tag{3.34}$$

where $d(Z/x)/dx$ is known from (3.24). From (3.26),

$$\frac{d}{dx}\left(\frac{Z}{x}\right) \sim \tfrac{3}{2}x^{-4} \quad \text{as } x \to \infty. \tag{3.35}$$

Therefore, subtracting $3/2$ from the integrand to ensure convergence, and adding $3r/2$ outside the integral,

$$\frac{1}{4r}\int^r x^4\frac{d}{dx}\left(\frac{Z}{x}\right)dx = \frac{3}{8} - \frac{H}{4r}, \tag{3.36}$$

where

$$H = \frac{3}{2}\int_r^\infty \{x^{7/2}(x^4-1)(x^3-1)^{-5/2} - 1\}\,dx. \tag{3.37}$$

Similarly,

$$R_2 = -\tfrac{1}{2}r^2 Z - \tfrac{1}{4}rG, \tag{3.38}$$

where

$$G = \frac{3}{2}\int_r^\infty x^{3/2}(x^4-1)(x^3-1)^{-5/2}\,dx. \tag{3.39}$$

As $r \to \infty$,

$$G \sim \tfrac{3}{2}r^{-1}, \tag{3.40}$$

$$H \sim \tfrac{15}{8}r^{-2}. \tag{3.41}$$

Therefore, as $r \to \infty$,

$$R = \frac{R_2 - R_1}{2} \sim -\tfrac{1}{2}. \tag{3.42}$$

The function $\psi_1$ is the sum of $R\sin\theta$ and the complementary function (3.9). Since $\psi$ must be constant on $r = 1$,

$$\psi_1 = \left\{R - \frac{R(1)}{r}\right\}\sin\theta + \tfrac{1}{2}a_0 + \sum_1^{\infty} \{a_n(r^n - r^{-n})\cos n\theta + c_n(r^n - r^{-n})\sin n\theta\}. \tag{3.43}$$

This solution cannot satisfy the boundary condition at infinity for any values of $a_n$, $c_n$. However, the part of the flow pattern in which deviations from the upstream conditions would be expected to be important is a strip

$$|y| = |r\sin\theta| < k, \tag{3.44}$$

where $k$ is finite. In principle, the complete flow pattern could be obtained by joining three solutions; the present one in the strip $|y| < k$, and two solutions for $|y| > k$ which would have to be found by another method. However, since the outer flows would be very nearly parallel, and would have little effect on the inner flow near $y = 0$, if $k$ were taken to be large enough, it should be sufficient to satisfy the boundary condition at infinity in the strip $|y| < k$ only. In this strip, the boundary condition is satisfied if $a_n$, $c_n$ are zero for all $n$, i.e. if

$$\begin{aligned}\psi_1 &= \left\{R - \frac{R(1)}{r}\right\}\sin\theta \\ &\sim -\frac{y}{2r} \quad \text{as } r \to \infty.\end{aligned} \tag{3.45}$$

*Third approximation to $\zeta$*

The equation for $g_2$ may be written in the form

$$g_2 = g_2'(r)\cos\theta + g_2''(r)\sin^2\theta\cos\theta, \tag{3.46}$$

where
$$g_2 = \tfrac{1}{2}r^{-4}Z - r^{-1}Z\,\frac{dQ}{dr} + \tfrac{3}{2}r^{-1/2}(r^3 - 1)^{-3/2}, \tag{3.47}$$

$$g_2'' = \left(r - \frac{1}{r^3}\right)\frac{dZ}{dr} - \left(r + \frac{1}{r^3}\right)\frac{Z}{r}.$$

When $n = 2$, a particular integral of (3.1) is

$$X(r) + Y(r)\sin^2\theta,$$

where
$$-\left(1 - \frac{1}{r^3}\right)\frac{dX}{dr} = \frac{3}{2r^4}X + g_2', \tag{3.48}$$

$$-\left(1-\frac{1}{r^3}\right)\frac{dY}{dr}+2\left(1+\frac{1}{2r^3}\right)\frac{Y}{r}=\frac{3}{2r^4}Y+g_2''. \tag{3.49}$$

Hence, solving these equations,

$$X=\left(\frac{r^3}{r^3-1}\right)^{1/2}X_1(r), \tag{3.50}$$

$$Y=r^{1/2}(r^3-1)^{1/2}Y_1(r), \tag{3.51}$$

where

$$X_1=\int_r^\infty\left(\frac{x^3}{x^3-1}\right)^{1/2}g_2'(x)\,dx, \tag{3.52}$$

$$Y_1=\int_r^\infty x^{5/2}(x^3-1)^{-3/2}g''(x)\,dx. \tag{3.53}$$

As $r\to\infty$,

$$X\sim-\tfrac{1}{4}r^{-4}, \tag{3.54}$$

$$Y\sim\tfrac{1}{2}r^{-1}. \tag{3.55}$$

Therefore, the solution of (3.1) for $\zeta_2$, satisfying $\zeta_2=0$ at $r=\infty$, is

$$\zeta_2=X(r)+Y(r)\sin^2\theta. \tag{3.56}$$

*Third approximation to $\psi$*

Substituting (3.56) in (3.8) gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_2}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi_2}{\partial\theta^2}=-X-\tfrac{1}{2}Y+\tfrac{1}{2}Y\cos 2\theta. \tag{3.57}$$

A particular integral is

$$S(r)+T(r)\cos 2\theta,$$

where

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dS}{dr}\right)=-X-\tfrac{1}{2}Y, \tag{3.58}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)-\frac{4T}{r^2}=\tfrac{1}{2}Y. \tag{3.59}$$

Integration of (3.58) gives, after manipulation similar to the previous cases,

$$S\doteq-\tfrac{1}{4}r-\int_r^\infty\{(X+\tfrac{1}{2}Y)x+\tfrac{1}{4}\}\log\left(\frac{x}{r}\right)dx. \tag{3.60}$$

The integration of (3.59) is carried out in the same manner as the integration of (3.31). Two integrals, $T_1$ and $T_2$, are defined by

$$\left.\begin{aligned}T_1&=r\frac{dT}{dr}-2T,\\ T_2&=r\frac{dT}{dr}+2T.\end{aligned}\right\} \tag{3.61}$$

These are found to be

$$T_1=\tfrac{1}{12}r+\tfrac{3}{8}r^{-2}\log r-r^{-2}\int_r^\infty\{\tfrac{1}{2}x^3Y-\tfrac{1}{4}x-\tfrac{3}{8}x^{-1}\}\,dx, \tag{3.62}$$

$$T_2=-\tfrac{1}{2}r^2\int_r^\infty x^{-1}Y\,dx \tag{3.63}$$

$$\sim-\tfrac{1}{4}r\quad\text{as } r\to\infty.$$

Hence, as $r \to \infty$,

$$T \doteq \frac{T_2 - T_1}{4} \sim -\tfrac{1}{12}r. \tag{3.64}$$

Again, it is not possible to satisfy the boundary condition at infinity for all values of $\theta$. However, the solution satisfying the condition $\psi_2 = 0$ at infinity in the strip $|y| < k$, as well as the condition $\psi_2 =$ constant on $r = 1$, is

$$\psi_2 = S + \left\{T - \frac{T(1)}{r^2}\right\}\cos 2\theta + \frac{1}{3}\left(r - \frac{1}{r}\right)\cos\theta \tag{3.65}$$

$$\sim \frac{y^4}{r^3} \quad \text{as } r \to \infty.$$

## 4. The velocity field

The results of § 3 are summarised below:

$$\frac{\zeta}{A} = \zeta_{00} + K\zeta_{11}\sin\theta + K^2\zeta_{20} + K^2\zeta_{22}\cos 2\theta + O(K^3), \tag{4.1}$$

$$\psi = \Psi_{00} + \Psi_{02}\cos 2\theta + K\Psi_{11}\sin\theta + K^2\Psi_{20} + \\ + K^2\Psi_{21}\cos\theta + K^2\Psi_{22}\cos 2\theta + O(K^3), \tag{4.2}$$

where

$$\left.\begin{aligned} \zeta_{00} &= \left(1 - \frac{1}{r^2}\right)^{-1/2}, & \zeta_{11} &= Z, \\ \zeta_{20} &= X + \tfrac{1}{2}Y, & \zeta_{22} &= -\tfrac{1}{2}Y, \\ \Psi_{00} &= -\tfrac{1}{4}r^2 - Q, & \Psi_{02} &= \frac{1}{4}\left(r^2 - \frac{1}{r^2}\right), \\ \Psi_{11} &= R - \frac{R(1)}{r}, & \Psi_{20} &= S, \\ \Psi_{21} &= \frac{1}{3}\left(r - \frac{1}{r}\right), & \Psi_{22} &= T - \frac{T(1)}{r^2}. \end{aligned}\right\} \tag{4.3}$$

The integrals $Q$, $R$, $S$, etc. have been evaluated and are given in table 1.

| $1/r$ | $Q$ | $-R$ | $-S$ | $T$ | $-X$ | $Y$ | $-Z$ |
|---|---|---|---|---|---|---|---|
| 0·0 | 0·0000 | 0·5000 | $\infty$ | $\infty$ | 0·0000 | 0·0000 | 0·0000 |
| 0·1 | 0·0500 | 0·4999 | 2·2005 | 0·8358 | 0·0000 | 0·0501 | 0·0050 |
| 0·2 | 0·1003 | 0·4994 | 1·2540 | 0·4329 | 0·0003 | 0·1011 | 0·0202 |
| 0·3 | 0·1502 | 0·4980 | 0·8373 | 0·2908 | 0·0014 | 0·1556 | 0·0464 |
| 0·4 | 0·2006 | 0·4953 | 0·6320 | 0·2274 | 0·0047 | 0·2177 | 0·0859 |
| 0·5 | 0·2515 | 0·4909 | 0·5108 | 0·1917 | 0·0125 | 0·2947 | 0·1436 |
| 0·6 | 0·3032 | 0·4841 | 0·4319 | 0·1691 | 0·0341 | 0·3999 | 0·2295 |
| 0·7 | 0·3562 | 0·4745 | 0·3773 | 0·1532 | 0·0821 | 0·5617 | 0·3659 |
| 0·8 | 0·4114 | 0·4609 | 0·3378 | 0·1401 | 0·2038 | 0·8573 | 0·6133 |
| 0·9 | 0·4701 | 0·4413 | 0·3081 | 0·1268 | 0·6450 | 1·6112 | 1·2219 |
| 1·0 | 0·5377 | 0·4089 | 0·2818 | 0·1077 | $\infty$ | $\infty$ | $\infty$ |

Table 1

The velocity components $u$ and $v$ are obtained from (2.5), (2.9) and (4.2), in the forms

$$\frac{u}{U} = -(1-r^{-3})\cos\theta - 2Kr^{-1}\Psi_{02}\sin 2\theta + K^2 r^{-1}\Psi_{11}\cos\theta - \\ - K^3\{r^{-1}\Psi_{21}\sin\theta + 2r^{-1}\Psi_{22}\sin 2\theta\} + O(K^4), \quad (4.4)$$

$$\frac{v}{U} = (1+\tfrac{1}{2}r^{-3})\sin\theta - K\left\{\frac{d\Psi_{00}}{dr} + \frac{d\Psi_{02}}{dr}\cos 2\theta\right\} - \\ - K^2\frac{d\Psi_{11}}{dr}\sin\theta - K^3\left\{\frac{d\Psi_{20}}{dr} + \frac{d\Psi_{21}}{dr}\cos\theta + \frac{d\Psi_{22}}{dr}\cos 2\theta\right\} + O(K^4). \quad (4.5)$$

The velocity is always finite, and this suggests that the series (4.4) and (4.5) converge when $K$ is small enough.

The position of the stagnation point is easily found from (4.5). On $r = 1$

$$\frac{v}{U} = 1\cdot5000\sin\theta - K\{0\cdot3505 + \cos 2\theta\} + 0\cdot9667K^2\sin\theta + \\ + K^3\{0\cdot5262 - 0\cdot6667\cos\theta + 0\cdot5524\cos 2\theta\} + O(K^4). \quad (4.6)$$

Put $$y = r\sin\theta. \quad (4.7)$$

Then, if $y_s$ is the value of $y$ at the stagnation point,

$$y_s = Ky_{1s} + K^3 y_{3s} + O(K^5), \quad (4.8)$$

and equation (4.6) gives

$$0 = K(1\cdot5000y_{1s} - 1\cdot3505) + \\ + K^3(1\cdot5000y_{3s} + 2\cdot0000y_{1s} + 0\cdot9667y_{1s} + 0\cdot4119) + O(K^5), \quad (4.9)$$

whence $$\left.\begin{aligned} y_{1s} &= \phantom{-}0\cdot9004, \\ y_{3s} &= -1\cdot9357. \end{aligned}\right\} \quad (4.10)$$

## 5. Streamlines and displacement effect

The streamlines are integrals of the equation

$$\frac{dr}{u} = \frac{rd\theta}{v}. \quad (5.1)$$

Substituting from (2.5), this can be written

$$K\left(\frac{\partial\psi}{\partial r}dr + \frac{\partial\psi}{\partial\theta}d\theta\right) = \frac{\partial\phi}{r\partial\theta}dr - \frac{\partial\phi}{dr}r\,d\theta, \quad (5.2)$$

or, using (2.9),

$$K\,d\psi = \frac{1}{2r\sin\theta}\,d\left\{\left(1-\frac{1}{r^3}\right)r^2\sin^2\theta\right\} \tag{5.3}$$

$$= \frac{1}{2y}\,dP,$$

where

$$P = \left(1-\frac{1}{r^3}\right)y^2 = py^2. \tag{5.4}$$

Using $y$ and $r$ as independent variables, (5.3) can be written

$$K\frac{\partial\psi}{\partial r}dr + K\frac{\partial\psi}{\partial y}dy = \frac{1}{2y}\frac{\partial P}{\partial r}dr + \frac{1}{2y}\frac{\partial P}{\partial y}dy, \tag{5.5}$$

i.e.

$$\left\{K\frac{\partial\psi}{\partial y} - \frac{1}{2y}\frac{\partial P}{\partial y}\right\}\frac{dy}{dr} + \left\{K\frac{\partial\psi}{\partial r} - \frac{1}{2y}\frac{\partial P}{\partial r}\right\} = 0, \tag{5.6}$$

which is a differential equation for $y$ as a function of $r$ along a streamline. Now, $\psi$ can be expanded in the form

$$\psi = \psi_{00} + \psi_{02}y^2 + K\psi_{11}y + \dots, \tag{5.7}$$

where the $\psi_{ij}$'s can be found by comparison with (4.2). Then, assuming that on a streamline

$$y = y_0(r) + Ky_1(r) + \dots, \tag{5.8}$$

a series of linear differential equations for $y_i(r)$ is obtained. These equations are

$$p\frac{dy_i}{dr} + \frac{1}{2}\frac{dp}{dr}y_i = n_i(r, y_0, \dots, y_{i-1}), \tag{5.9}$$

where

$$\left.\begin{aligned} n_0 &= 0, \\ n_1 &= \frac{\partial\psi_{00}}{\partial r} + \frac{\partial\psi_{02}}{\partial r}y_0^2 + 2\psi_{02}\,y_0\frac{dy_0}{dr}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{5.10}$$

The solutions of (5.9) are

$$y_0 = C_0\,p^{-1/2}, \tag{5.11}$$

$$y_i = -p^{-1/2}\int_r^\infty p^{-1/2}n_i\,dr + C_i\,p^{-1/2}. \tag{5.12}$$

It can be seen that (5.11) is the equation for the streamlines in the irrotational flow, as would be expected from (5.8).

The streamline reaching the stagnation point is of particular interest since it gives the displacement effect. On this streamline,

$$y_0 = y_2 = y_4 = \dots = 0. \tag{5.13}$$

Hence

$$\left.\begin{aligned} n_1 &= \frac{\partial\psi_{00}}{\partial r}, \\ n_3 &= \frac{\partial\psi_{02}}{\partial r}y_1^2 + \frac{\partial\psi_{11}}{\partial r}y_1 + \frac{\partial\psi_{20}}{\partial r} + (2\psi_{02}y_1 + \psi_{11})\frac{dy_1}{dr}, \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{5.14}$$

Since $y_i$ must be finite when $r = 1$, the coefficient of $p^{-1/2}$ in (5.12) must be zero, i.e.

$$C_i = \int_1^\infty p^{-1/2} n_i \, dr. \tag{5.15}$$

Hence

$$y_i = p^{-1/2} \int_1^r p^{-1/2} n_i \, dr. \tag{5.16}$$

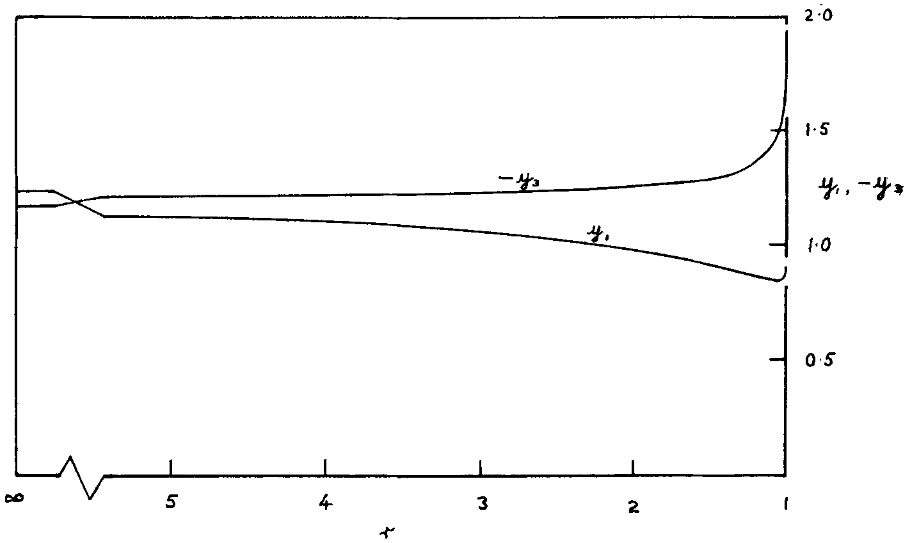


Figure 3. The stagnation steamline, $y(r) = Ky_1(r) + K^3y_3(r) + O(K^5)$.

Using (5.14), the functions $y_i$ may now be evaluated. The results for $y_1$ and $y_3$ are shown graphically in figure 3.

The displacement, $\delta$, of the stagnation streamline at infinity is given by

$$\begin{aligned} \frac{\delta}{a} &= Ky_1(\infty) + K^3y_3(\infty) + O(K^5) \qquad (5.17) \\ &= 1{\cdot}2400K - 1{\cdot}1752K^3 + O(K^5). \end{aligned}$$

The variation of $\delta/2a$ with $K$ is shown in figure 4. It can be seen that the displacement effect found in this investigation is of the same order of magnitude as the experimental result of Young & Maas. The gradient of $\delta/2a$ with $K$, when $K$ is small, is somewhat less than would be desired for

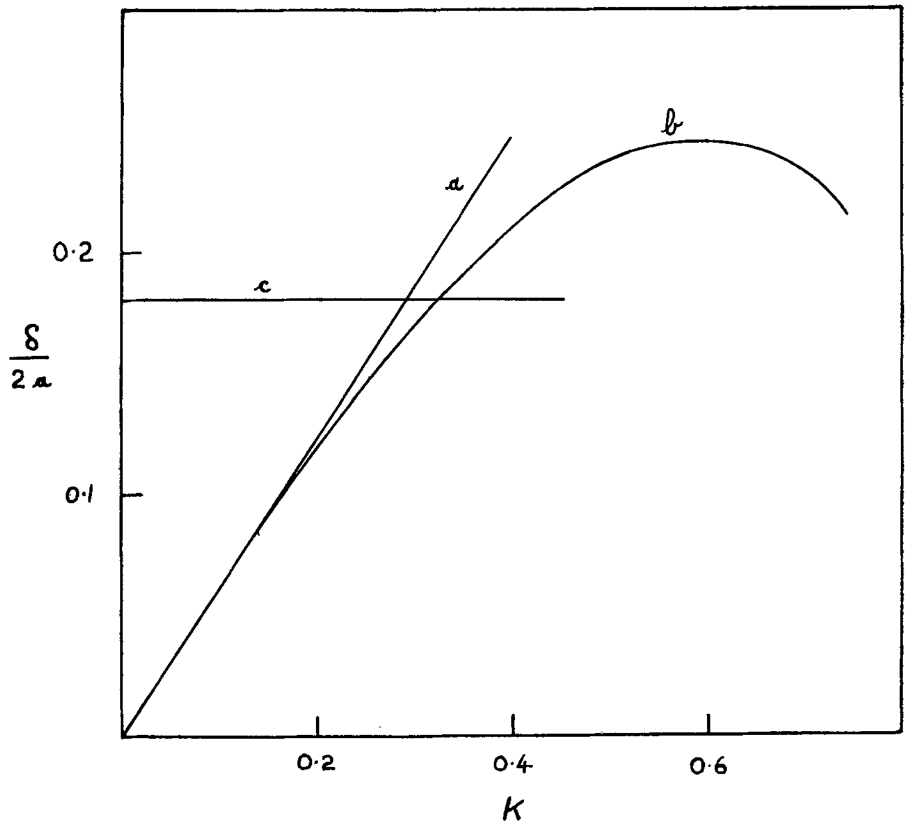


Figure 4. The displacement effect of a sphere: ($a$) first approximation neglecting $O(K^3)$; ($b$) next approximation neglecting $O(K^5)$; ($c$) experiment, flat-nosed pitot tube.

good agreement with experiment. However, as indicated in § 1, the displacement effect of a sphere is probably the same as that of a pitot tube of smaller diameter. Therefore, the initial gradient of $\delta/2a$ would be expected to be larger for a conventional pitot tube than for a sphere.

It may be concluded from the above results that the major part of the displacement effect of a pitot tube in a shear flow may be accounted for without the inclusion of viscosity; i.e. the vorticity field is of prime importance.

This work was done while the author was at the Fluid Motion Laboratory, University of Manchester. The author wishes to thank Professor M. J. Lighthill for suggesting the problem and for his continued interest and advice during the preparation of this paper.

## Appendix. Shear flow past two-dimensional bodies

The flow pattern around a two dimensional body in a flow with constant rate of shear $A$ is defined by a stream function $\psi$ which is a solution of

$$\nabla^2\psi = -A, \tag{A. 1}$$

subject to the condition that $\psi$ is constant on the surface of the body (e.g. Milne-Thomson 1949, p. 108). Since $\psi$ is constant on any streamline the displacement $\delta$ of the stagnation streamline is obtained directly from the value of $\psi$ on the surface of the body.

Solutions of (A. 1), and hence $\delta$, are given below for a circular cylinder and for a parallel plate pitot tube.

### (*a*) *The circular cylinder*

The centre of a circular cylinder of radius $a$ is taken as the origin of coordinates $r$, $\theta$, and the undisturbed flow has velocity $U+Ar\sin\theta$ in the direction $\theta = \pi$. The stream function for this flow is (Lamb 1932, Tsien 1943, Reichardt 1954)

$$\psi = -\tfrac{1}{4}A(r^2-a^2)+\tfrac{1}{4}A\left(r^2-\frac{a^4}{r^2}\right)\cos 2\theta - U\left(r-\frac{a^2}{r}\right)\sin\theta. \tag{A. 2}$$

Since $\psi$ is zero on the cylinder, the displacement $\delta$ is the value of $r\sin\theta$ for which $\psi$ is zero at infinity, so that

$$\tfrac{1}{4}Aa^2-\tfrac{1}{2}A\delta^2-U\delta = 0. \tag{A. 3}$$

Hence

$$\frac{\delta}{a} = K^{-1}\{-1+\surd(1+\tfrac{1}{2}K^2)\} \tag{A. 4}$$

$$\sim \tfrac{1}{4}K \quad \text{as } K\to 0. \tag{A. 5}$$

The negative square root has been discarded in (A. 4) because $\delta$ must be zero when $K$ is zero. The variation of $\delta/2a$ with $K$ is shown in figure 5.

An interesting deduction about the stagnation streamline can be made from (A. 2). The term $-\tfrac{1}{4}Aa^4r^{-2}\cos 2\theta$ represents the effects of the image vorticity inside the cylinder, which tends to deflect the oncoming flow downwards. But the doublet term $Ua^2r^{-1}\sin\theta$ tends to deflect the flow away from the axis. Hence, the stagnation streamline is the one on which these effects balance, i.e. the one on which

$$\frac{Ua^2}{r}\sin\theta - \frac{Aa^4}{4r^2}\cos 2\theta = 0. \tag{A. 6}$$

Therefore, if $\theta$ is always small on the stagnation streamline, the latter is straight and parallel to the axis at a distance $r\sin\theta$ from it, where

$$\frac{a}{r}\sin\theta = \tfrac{1}{4}K, \tag{A. 7}$$

in agreement with (A. 5). This result can also be obtained mathematically from (A. 2).

(*b*) *The parallel-plate pitot tube*

This idealized pitot tube consists of two semi-infinite flat plates at $y = \pm c$, stretching from $x = 0$ to $x = \infty$, with no net mass flow between the plates (cf. Lamb 1932, p. 75). The undisturbed flow has velocity in the positive $x$-direction.

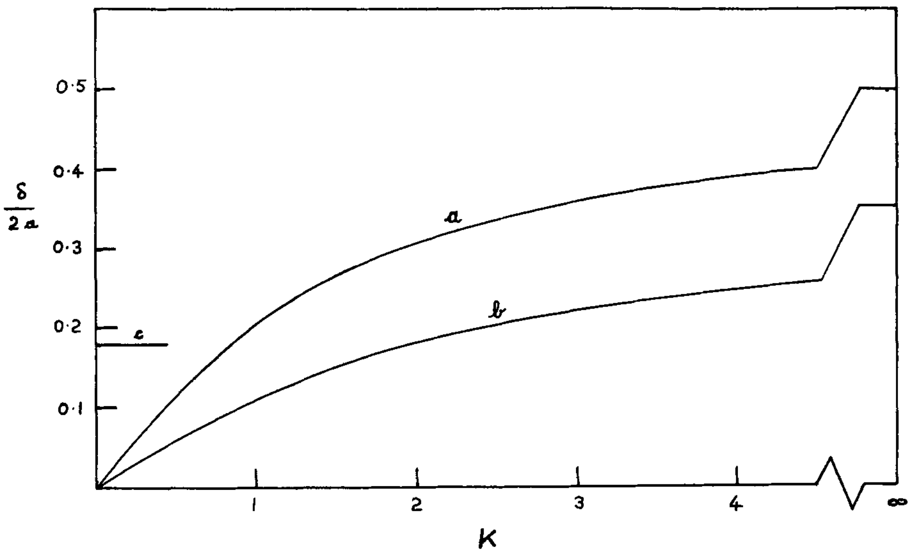


Figure 5. The displacement effect of two-dimensional bodies: (*a*) parallel plate pitot tube, distance $2a$ apart; (*b*) circular cylinder; (*c*) experimental result for flat-nosed axially symmetric pitot tube.

Let the stream function be $\psi_0$ when $A$ is zero, where $\psi_0$ is zero on the surface of the body. At infinity, $\psi_0 \sim Uy$. Then, in the general case, the stream function is

$$\psi = \psi_0 + \tfrac{1}{2}A. \tag{A. 8}$$

The displacement $\delta$ must satisfy the equation

$$U\delta + \tfrac{1}{2}A(\delta^2 - c^2) = 0, \tag{A. 9}$$

whence

$$\frac{\delta}{c} = K^{-1}\{-1 + \sqrt{(1+K^2)}\} \tag{A. 10}$$

$$\sim \tfrac{1}{2}K \quad \text{as } K \to 0. \tag{A. 11}$$

Comparison of (A. 9) with (A. 3) shows that, for identical oncoming flows, the displacement effect of the cylinder is equal to that of the parallel plate pitot tube when

$$c = \frac{a}{\sqrt{2}}, \tag{A. 12}$$

i.e. when the diameter of the cylinder is $\sqrt{2}$ times the spacing of the plates.

Figure 5 shows that $\delta/2c$ always increases with $K$ and only reaches 0·18 when $K = 0{\cdot}83$. Since the highest value of $K$ used in the experiments of Young & Maas was 0·3, this solution does not afford an explanation of the displacement effect.

The above analysis is only slightly modified when the two-dimensional pitot tube is of arbitrary shape, provided that it consists of two parallel plates at infinity. These are at a distance $2c$ apart as before. In this case, the stream function is

$$\psi = \psi_0 + \tfrac{1}{2}Ay^2 + \psi_1, \tag{A. 13}$$

where $\psi_0$ is the stream function when $A$ is zero, and $\psi_1$ is an irrotational stream function included to satisfy the boundary conditions. But, on the body, $\psi_1$ can differ from $-\frac{1}{2}Ac^2$ over a finite region only. Hence it must always take this value on the body, and the displacement effect is the same as before.

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